4.6 Gradient, Divergence, Curl and Laplacian

In this final section we will establish some relationships between the gradient, divergence and curl, and we will also introduce a new quantity called the *Laplacian*. We will then show how to write these quantities in cylindrical and spherical coordinates.

For a real-valued function $f(x,y,z)$ on $\mathbb{R}^3$, the gradient $\nabla f(x,y,z)$ is a vector-valued function on $\mathbb{R}^3$, that is, its value at a point (x,y,z) is the vector

in $\mathbb{R}^3$, where each of the partial derivatives is evaluated at the point (x,y,z). So in this way, you can think of the *symbol* ∇ as being "applied" to a real-valued function f to produce a vector ∇f.

It turns out that the divergence and curl can also be expressed in terms of the symbol ∇. This is done by thinking of ∇ as a *vector* in $\mathbb{R}^3$, namely

$$\nabla \;=\; \frac{\partial}{\partial x}\mathbf{i} \;+\; \frac{\partial}{\partial y}\mathbf{j} \;+\; \frac{\partial}{\partial z}\mathbf{k}\,. \tag{4.51}$$

Here, the symbols $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ are to be thought of as "partial derivative operators" that will get "applied" to a real-valued function, say $f(x,y,z)$, to produce the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$. For instance, $\frac{\partial}{\partial x}$ "applied" to $f(x,y,z)$ produces $\frac{\partial f}{\partial x}$.

Is ∇ *really* a vector? Strictly speaking, no, since $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ are not actual numbers. But it helps to *think* of ∇ as a vector, especially with the divergence and curl, as we will soon see. The process of "applying" $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ to a real-valued function $f(x,y,z)$ is normally thought of as *multiplying* the quantities:

$$\left(\frac{\partial}{\partial x}\right)(f) = \frac{\partial f}{\partial x}\,, \quad \left(\frac{\partial}{\partial y}\right)(f) = \frac{\partial f}{\partial y}\,, \quad \left(\frac{\partial}{\partial z}\right)(f) = \frac{\partial f}{\partial z}$$

For this reason, ∇ is often referred to as the "del operator", since it "operates" on functions.

For example, it is often convenient to write the divergence div $\mathbf{f}$ as $\nabla\cdot\mathbf{f}$, since for a vector field $\mathbf{f}(x,y,z) = f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k}$, the dot product of $\mathbf{f}$ with ∇ (thought of as a vector) makes sense:

$$\begin{aligned} \nabla\cdot\mathbf{f} &= \\ &= \\ &= \\ &= \end{aligned}$$

We can also write curl $\mathbf{f}$ in terms of ∇, namely as $\nabla \times \mathbf{f}$, since for a vector field $\mathbf{f}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$, we have:

$$\nabla \times \mathbf{f} =$$

$$=$$

$$=$$

$$=$$

For a real-valued function $f(x,y,z)$, the gradient $\nabla f(x,y,z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$ is a vector field, so we can take its divergence:

$$\text{div}\,\nabla f = \nabla \cdot$$

$$=$$

$$=$$

$$=$$

Note that this is a real-valued function, to which we will give a special name:

Definition 4.7. For a real-valued function $f(x,y,z)$, the **Laplacian** of f, denoted by Δf, is given by

$$\Delta f(x,y,z) = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\,. \tag{4.52}$$

Often the notation $\nabla^2 f$ is used for the Laplacian instead of Δf, using the convention $\nabla^2 = \nabla \cdot \nabla$.

Example 4.17. Let $\mathbf{r}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ be the position vector field on $\mathbb{R}^3$. Then $\|\mathbf{r}(x,y,z)\|^2 = \mathbf{r}\cdot\mathbf{r} = x^2 + y^2 + z^2$ is a real-valued function. Find

(a) the gradient of $\|\mathbf{r}\|^2$

(b) the divergence of $\mathbf{r}$

(c) the curl of $\mathbf{r}$

(d) the Laplacian of $\|\mathbf{r}\|^2$

Solution: (a) $\nabla \|\mathbf{r}\|^2 = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} = 2\,\mathbf{r}$

(b) $\nabla \cdot \mathbf{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$

(c)

$$\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (0-0)\,\mathbf{k} = \mathbf{0}$$

(d) $\Delta \|\mathbf{r}\|^2 = \frac{\partial^2}{\partial x^2}(x^2+y^2+z^2) + \frac{\partial^2}{\partial y^2}(x^2+y^2+z^2) + \frac{\partial^2}{\partial z^2}(x^2+y^2+z^2) = 2 + 2 + 2 = 6$

Note that we could have calculated $\Delta \|\mathbf{r}\|^2$ another way, using the ∇ notation along with parts (a) and (b):

$$\Delta \|\mathbf{r}\|^2 = \nabla \cdot \nabla \|\mathbf{r}\|^2 = \nabla \cdot 2\mathbf{r} = 2\nabla \cdot \mathbf{r} = 2(3) = 6$$

Notice that in Example 4.17 if we take the curl of the gradient of $\|\mathbf{r}\|^2$ we get

$$\nabla \times (\nabla \|\mathbf{r}\|^2) = \nabla \times 2\mathbf{r} = 2\nabla \times \mathbf{r} = 2\mathbf{0} = \mathbf{0}\,.$$

The following theorem shows that this will be the case in general:

Theorem 4.15. For any smooth real-valued function $f(x,y,z)$, $\nabla \times (\nabla f) = \mathbf{0}$.

Proof: We see by the smoothness of f that

since the mixed partial derivatives in each component are equal. **QED**

Corollary 4.16. If a vector field $\mathbf{f}(x,y,z)$ has a potential, then curl $\mathbf{f} = \mathbf{0}$.

Another way of stating Theorem 4.15 is that gradients are irrotational. Also, notice that in Example 4.17 if we take the divergence of the curl of $\mathbf{r}$ we trivially get

$$\nabla \cdot (\nabla \times \mathbf{r}) = \nabla \cdot \mathbf{0} = 0\,.$$

The following theorem shows that this will be the case in general:

Theorem 4.17. For any smooth vector field $\mathbf{f}(x,y,z)$, $\nabla \cdot (\nabla \times \mathbf{f}) = 0$.

The proof is straightforward and left as an exercise for the reader.

Corollary 4.18. The flux of the curl of a smooth vector field $\mathbf{f}(x,y,z)$ through any closed surface is zero.

Proof: Let Σ be a closed surface which bounds a solid S. The flux of $\nabla \times \mathbf{f}$ through Σ is

$$\begin{aligned}\iint_\Sigma (\nabla \times \mathbf{f}) \cdot d\boldsymbol{\sigma} &= \iiint_S \nabla \cdot (\nabla \times \mathbf{f})\, dV \quad \text{(by the Divergence Theorem)} \\ &= \iiint_S 0\, dV \quad \text{(by Theorem 4.17)} \\ &= 0\,.\end{aligned}$$

QED

There is another method for proving Theorem 4.15 which can be useful, and is often used in physics. Namely, if the surface integral $\iint_\Sigma f(x,y,z)\,d\sigma = 0$ for *all* surfaces Σ in some solid region (usually all of $\mathbb{R}^3$), then we must have $f(x,y,z) = 0$ throughout that region. The proof is not trivial, and physicists do not usually bother to prove it. But the result is true, and can also be applied to double and triple integrals.

For instance, to prove Theorem 4.15, assume that $f(x,y,z)$ is a smooth real-valued function on $\mathbb{R}^3$. Let C be a simple closed curve in $\mathbb{R}^3$ and let Σ be any capping surface for C (i.e. Σ is orientable and its boundary is C). Since ∇f is a vector field, then

$$\begin{aligned}\iint_\Sigma (\nabla \times (\nabla f)) \cdot \mathbf{n}\, d\sigma &= \oint_C \nabla f \cdot d\mathbf{r} \quad \text{by Stokes' Theorem, so} \\ &= 0 \quad \text{by Corollary 4.13.}\end{aligned}$$

Since the choice of Σ was arbitrary, then we must have $(\nabla \times (\nabla f)) \cdot \mathbf{n} = 0$ throughout $\mathbb{R}^3$, where $\mathbf{n}$ is any unit vector. Using $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in place of $\mathbf{n}$, we see that we must have $\nabla \times (\nabla f) = \mathbf{0}$ in $\mathbb{R}^3$, which completes the proof.

Example 4.18. A system of electric charges has a *charge density* $\rho(x,y,z)$ and produces an electrostatic field $\mathbf{E}(x,y,z)$ at points (x,y,z) in space. *Gauss' Law* states that

$$\iint_\Sigma \mathbf{E} \cdot d\boldsymbol{\sigma} = 4\pi \iiint_S \rho\, dV$$

for any closed surface Σ which encloses the charges, with S being the solid region enclosed by Σ. Show that $\nabla \cdot \mathbf{E} = 4\pi\rho$. This is one of *Maxwell's Equations*.[10]

[10] In Gaussian (or CGS) units.